Simple Restricted Modules For the Restricted Contact Lie Algebras

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Abstract

It is shown that, with a few exceptions, the simple restricted modules for a restricted contact Lie algebra are induced from those for the homogeneous component of degree zero.

In [3], Shen constructed the simple restricted modules for the restricted Witt, special and hamiltonian Lie algebras which comprise three of the four classes of restricted Lie algebras of Cartan type. His methods, however, do not apply to the algebras in the fourth class, namely, the contact algebras.

Here, this remaining case is considered and it is shown that, with a few exceptions, the simple restricted modules are induced from simple restricted modules (extended trivially to positive components) for the homogeneous component of degree zero in the usual grading of the algebra. As this component is isomorphic to the direct sum of a symplectic algebra and the trivial algebra, the problem of determining, say, the dimensions of the simple restricted modules is then reduced to the classical situation for which Lusztig has a conjecture (see [3], p. 294).

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1 Notation and Statement of Main Theorem

The notation will be, for the most part, the same as that in [4] and this reference can also be consulted for the precise definition and fundamental properties of the contact Lie algebras.

Let F be an algebraically closed field of characteristic $p > 2$ and let $n = 2r + 1$ with $r \in \mathbf{N}$. For $1 \leq k \leq n$ let ε_k be the n-tuple with jth component δ_{jk} (Kronecker delta). Set $A = \{a = \sum_k a_k \varepsilon_k \mid 0 \leq a_k < p\} \subset \mathbf{Z}^n$.

The underlying vector space of the restricted contact Lie algebra K (denoted $K(2r+1,\underline{1})$ in [4]) has as basis $\{x^{(a)} \mid a \in A\}$ if $n+3 \not\equiv 0 \pmod p$ and $\{x^{(a)} \mid a \in A,\ a \neq \sum(p-1)\varepsilon_k\}$ otherwise. (The $x^{(a)}$ are standard basis vectors for a divided power algebra.)

For $1 \leq k \leq 2r$, set

$$\sigma(k) = \begin{cases} 1, & 1 \leq k \leq r \\ -1, & r < k \leq 2r \end{cases}$$

and $k' = k + \sigma(k)r$. Also, for $a \in A$, set $|a| = \sum a_k$ and $\|a\| = |a| + a_n - 2$. With $\langle x, y\rangle$ denoting the Lie product of $x, y \in K$, the following relations hold. (By convention, $x^{(a)} = 0$ if $a \notin A$.)

Theorem 1.1. *1.* $\langle x^{(0)}, x^{(a)}\rangle = 2x^{(a-\varepsilon_n)}$.

2. $\langle x^{(\varepsilon_k)}, x^{(a)}\rangle = \sigma(k)x^{(a-\varepsilon_{k'})} + (a_k+1)x^{(a+\varepsilon_k-\varepsilon_n)}, \quad 1 \leq k \leq 2r.$

3. $\langle x^{(\varepsilon_n)}, x^{(a)}\rangle = \|a\|x^{(a)}$.

4. $\langle x^{(\varepsilon_k+\varepsilon_{k'})}, x^{(a)}\rangle = \sigma(k)(a_{k'} - a_k)x^{(a)}, \quad 1 \leq k \leq 2r.$

K acquires the structure of a graded Lie algebra if one defines the ith homogeneous component by $K_i = \{x^{(a)} \mid \|a\| = i\}$.

Denote the restricted universal enveloping algebra (u-algebra) of an arbitrary restricted Lie algebra L by $u(L)$. The category of (unitary) $u(L)$-modules and that of restricted L-modules are equivalent so, for our purposes, it suffices to work in the former category.

Let $\Lambda = \{\lambda = \sum_{i=1}^r \lambda_i\varepsilon_i + \lambda_n\varepsilon_n \mid \lambda_i, \lambda_n \in \mathbf{F}_p\}$ the elements of which are called *weights*. For a $u(K_0)$-module V and $\lambda \in \Lambda$ define $V_\lambda = \{v \in V \mid x^{(\varepsilon_i+\varepsilon_{i'})}v = \lambda_i v\ (1 \leq i \leq r) \text{ and } x^{(\varepsilon_n)}v = \lambda_n v\}$. A nonzero vector $m \in V_\lambda$ is a *maximal vector* (of weight λ) if $x^{(\varepsilon_i+\varepsilon_j)}m = 0$ for all $(i,j) \in I := \{(i,j) \mid 1 \leq i,j \leq r \text{ or } 1 \leq i \leq r,\ i' < j \leq 2r\}$.

For each $\lambda \in \Lambda$ there exists a simple $u(K_0)$-module $S(\lambda)$ possessing a unique (up to scalar multiples) maximal vector of weight λ. Moreover, $\{S(\lambda) \mid \lambda \in \Lambda\}$ is a complete set of representatives for the isomorphism classes of simple $u(K_0)$-modules. In fact, K_0 is the direct sum of its ideals $\sum_{1\leq i,j,\leq 2r} Fx^{(\varepsilon_i+\varepsilon_j)} \cong \mathfrak{sp}(2r)$ and $Fx^{(\varepsilon_n)} \cong F$ and it is easy to see that $S(\lambda)$ is a simple $u(\mathfrak{sp}(2r))$-module on which $x^{(\varepsilon_n)}$ acts as multiplication by λ_n, so classical theory applies.

Set $N^+ = \sum_{i>0} K_i$. Then $N^+ \lhd N^+ + K_0 =: K^+$ and $K^+/N^+ \cong K_0$. In particular, any K_0-module becomes a K^+-module in the natural way. It follows from [1] that if S is a simple $u(K_0)$-module, then the $u(K)$-module $M(S) := u(K) \otimes_{u(K^+)} S$ has a unique simple quotient and that this sets up a one-to-one correspondence between the simple $u(K_0)$-modules and the simple $u(K)$-modules.

For $1 \leq k \leq r+1$ set $\zeta_k = -\sum_{i=1}^{r-k+1} \varepsilon_i$ (the empty sum being zero). A weight $\lambda \in \Lambda$ is *exceptional* if $\lambda = \zeta_k + (\pm k - r - 1)\varepsilon_n$ for some k $(1 \leq k \leq r+1)$. The main result of the paper is as follows.

Theorem 1.2. *If $\lambda \in \Lambda$ is not exceptional, then $M(S(\lambda))$ is simple.*

The proof of this theorem is given in section 3. The converse has been checked by the author in the case $n = 3$ and it is conjectured to hold in general. Indeed, similar "exceptional" weights (i.e., ones for which the corresponding induced module is not simple) arose in Shen's findings for the other three classes of algebras, although there they turned out to be fundamental dominant weights because of a difference of indexing.

2 Lemmas

In this section, formulas will be obtained for use in the proof of 1.2.

Set

$$T_k = \begin{cases} x^{(\varepsilon_k)}, & 1 \leq k < n \\ x^{(0)}, & k = n, \end{cases}$$

and for $\beta = (\beta_k) \in \mathbf{Z}^n$ define $T^\beta = \prod_{k=1}^n T_k^{\beta_k} \in u(K)$ where $T_k^i := 0$ if $i < 0$. It is a consequence of the p-mapping defined on K that $T_k^i = 0$ if $i \geq p$, so that $T^\beta = 0$ if and only if $\beta \notin A$.

Lemma 2.1. *In each of the following situations, $T_k T^\beta = T^\beta T_k$.*

1. *$k = n$.*
2. *$k < n$ and $\beta_{k'} = 0$.*
3. *$\beta_n = p - 1$.*

Proof. Parts (1) and (2) are immediate from 1.1(1) and (2). Part (3) follows from part (1) and 1.1(2). □

In the following lemmas, fix $\lambda \in \Lambda$ and let m denote a maximal vector in $S(\lambda)$. For any $\beta \in A$, we have $T^\beta \otimes m \in u(K) \otimes_{u(K^+)} S(\lambda) = M(S(\lambda))$.

Lemma 2.2. *If $a_i, a_j \geq 1$ and $\beta_{i'} = 0 = \beta_{j'}$ for some $(i, j) \in I$, then $x^{(a)} T^\beta \otimes m = 0$.*

Proof. If $|\beta| = 0$, then $x^{(a)}T^\beta \otimes m = 1 \otimes x^{(a)}m = 0$, since m is a maximal vector. Now suppose $|\beta| > 0$ and let k be the least index for which $\beta_k \neq 0$. If $k = n$, then by 1.1,

$$x^{(a)}T^\beta \otimes m = T_n x^{(a)} T^{\beta-\varepsilon_n} \otimes m - 2x^{(a-\varepsilon_n)}T^{\beta-\varepsilon_n} \otimes m$$

which equals zero by the induction hypothesis. If $k < n$, then

$$\begin{aligned} x^{(a)}T^\beta \otimes m = {} & T_k x^{(a)} T^{\beta-\varepsilon_k} \otimes m - \sigma(k) x^{(a-\varepsilon_{k'})} T^{\beta-\varepsilon_k} \otimes m \\ & - (a_k+1) x^{(a+\varepsilon_k-\varepsilon_n)} T^{\beta-\varepsilon_k} \otimes m. \end{aligned}$$

If $k' \in \{i, j\}$, then $T^{\beta-\varepsilon_k} = 0$ and the result follows. Otherwise, the induction hypothesis applies to each of the three terms also giving the result. □

Below, $\binom{i}{j}$ denotes the binomial coefficient with the convention that $\binom{i}{j} = 0$ if either $i < j$ or $j < 0$.

Lemma 2.3. *$x^{(\varepsilon_k+\omega\varepsilon_n)}T_n^\mu \otimes m = (-2)^\omega \binom{\mu}{\omega} T_k T_n^{\mu-\omega} \otimes m$ for any $k < n$ and $\mu, \omega \in \mathbf{Z}$.*

Proof. If $\mu \leq 0$, then both sides equal zero unless $\mu = 0 = \omega$ in which case both sides equal $T_k \otimes m$. If $\mu > 0$, then by 1.1,

$$x^{(\varepsilon_k+\omega\varepsilon_n)}T_n^\mu \otimes m = T_n x^{(\varepsilon_k+\omega\varepsilon_n)} T_n^{\mu-1} \otimes m - 2x^{(\varepsilon_k+(\omega-1)\varepsilon_n)}T_n^{\mu-1} \otimes m$$

and the induction hypothesis applies to both terms on the right to give the result. □

Lemma 2.4. *If $\beta_j = 0$ for all $r < j < n$, then for each $\omega \in \mathbf{Z}$,*

$$x^{(\omega\varepsilon_n)}T^\beta \otimes m = \big[(-2)^{\omega-1}\binom{\beta_n}{\omega-1}(\lambda_n - \sum_{1\leq i\leq r}\beta_i) + (-2)^\omega\binom{\beta_n}{\omega}\big] T^{\beta-(\omega-1)\varepsilon_n} \otimes m.$$

Proof. If $|\beta| = 0$, then both sides equal zero if $\omega < 0$ or if $\omega > 1$ while both sides equal $T_n \otimes m$ if $\omega = 0$ and both sides equal $\lambda_n \cdot 1 \otimes m$ if $\omega = 1$. Now assume $|\beta| > 0$ and let k be the least index for which $\beta_k \neq 0$. If $k = n$, then the result follows from the induction hypothesis as in the proof of 2.2. If $k < n$ (so that, in fact, $k \leq r$), then

$$x^{(\omega\varepsilon_n)}T^\beta \otimes m = T_k x^{(\omega\varepsilon_n)} T^{\beta-\varepsilon_k} \otimes m - x^{(\varepsilon_k+(\omega-1)\varepsilon_n)}T^{\beta-\varepsilon_k} \otimes m.$$

But

$$x^{(\varepsilon_k+(\omega-1)\varepsilon_n)}T^{\beta-\varepsilon_k}\otimes m = T^{\beta-\varepsilon_k-\beta_n\varepsilon_n}x^{(\varepsilon_k+(\omega-1)\varepsilon_n)}T_n^{\beta_n}\otimes m$$
$$= (-2)^{\omega-1}\binom{\beta_n}{\omega-1}T^{\beta-(\omega-1)\varepsilon_n}\otimes m,$$

where the first equality is from 1.1 and 2.2 and the second equality is from 2.3. The result now follows from the induction hypothesis. □

Lemma 2.5. $x^{(\varepsilon_n)}T^\beta\otimes m = (\lambda_n - \sum_{k<n}\beta_k - 2\beta_n)T^\beta\otimes m.$

Proof. Use induction on $|\beta|$ as in the proof of 2.2. □

Lemma 2.6. *For* $1\le i\le r$, $x^{(\varepsilon_i+\varepsilon_{i'})}T^\beta\otimes m = (\beta_{i'}-\beta_i+\lambda_i)T^\beta\otimes m.$

Proof. Use induction on $|\beta|$ as in the proof of 2.2. □

Lemma 2.7. *If* $1\le i\le r$ *and* $\beta_k=0$ *for all* $r<k<n$, *then*

$$x^{(\varepsilon_i+\varepsilon_{i'}+\varepsilon_n)}T^\beta\otimes m = 2\beta_n(\beta_i-\lambda_i)T^{\beta-\varepsilon_n}\otimes m.$$

Proof. If $|\beta|=0$, then both sides equal zero, so suppose $|\beta|>0$ and let k be the least index for which $\beta_k\neq 0$. If $k=n$, the result follows easily from 1.1, 2.6 and the induction hypothesis. If $k<n$ (so that, in fact, $k\le r$), then by 1.1,

$$x^{(\varepsilon_i+\varepsilon_{i'}+\varepsilon_n)}T^\beta\otimes m = T_k x^{(\varepsilon_i+\varepsilon_{i'}+\varepsilon_n)}T^{\beta-\varepsilon_k}\otimes m - \delta_{ik}x^{(\varepsilon_i+\varepsilon_n)}T^{\beta-\varepsilon_k}\otimes m$$
$$-(\delta_{ik}+1)x^{(\varepsilon_i+\varepsilon_{i'}+\varepsilon_k)}T^{\beta-\varepsilon_k}\otimes m.$$

Now by 1.1 and 2.2,

$$x^{(\varepsilon_i+\varepsilon_n)}T^{\beta-\varepsilon_k}\otimes m = T^{\beta-\varepsilon_k-\beta_n\varepsilon_n}x^{(\varepsilon_i+\varepsilon_n)}T_n^{\beta_n}\otimes m,$$

while

$$x^{(\varepsilon_i+\varepsilon_{i'}+\varepsilon_k)}T^{\beta-\varepsilon_k}\otimes m = 0.$$

Hence, the induction hypothesis and 2.3 together give the result. □

Lemma 2.8. *If* $1\le i,k\le r$, $\beta_k=p-1$ *and* $\beta_{k'}=0$, *then* $x^{(\varepsilon_i+\varepsilon_k)}T^\beta\otimes m=0.$

Proof. If $\beta_{i'}=0$, then the assertion follows from 2.2. Assume $\beta_{i'}>0$ (so that, in particular, $i\neq k$). By 1.1,

$$x^{(\varepsilon_i+\varepsilon_k)}T^\beta\otimes m = T^{\beta-\beta_k\varepsilon_k-\beta_{i'}\varepsilon_{i'}}x^{(\varepsilon_i+\varepsilon_k)}T_{i'}^{\beta_{i'}}T_k^{\beta_k}\otimes m.$$

Now

$$x^{(\varepsilon_i+\varepsilon_k)}T_{i'}^{\beta_{i'}}T_k^{\beta_k}\otimes m = T_{i'}x^{(\varepsilon_i+\varepsilon_k)}T_{i'}^{\beta_{i'}-1}T_k^{\beta_k}\otimes m + x^{(\varepsilon_k)}T_{i'}^{\beta_{i'}-1}T_k^{\beta_k}\otimes m.$$

The first term on the right equals zero by the induction hypothesis and that the second term equals zero is a consequence of 2.1 and the assumption $\beta_k = p-1$. □

Lemma 2.9. *If $r < j < n$ and* $\beta_k = \begin{cases} p-1, & 1 \le k \le r \\ 0, & r < k < j \\ p-1, & j < k \le n, \end{cases}$ *then*

$$x^{(\varepsilon_{j'}+\varepsilon_n)}T^\beta \otimes m = \beta_j(\lambda_n - \lambda_{j'} - \beta_j - j + 3r + 2)T^{\beta-\varepsilon_j}\otimes m.$$

Proof. If $\beta_j = 0$, then 1.1, 2.8 and 2.2 imply

$$x^{(\varepsilon_{j'}+\varepsilon_n)}T^\beta \otimes m = T^{\beta-\beta_n\varepsilon_n}x^{(\varepsilon_{j'}+\varepsilon_n)}T_n^{\beta_n}\otimes m.$$

But by 2.3,

$$x^{(\varepsilon_{j'}+\varepsilon_n)}T_n^{\beta_n}\otimes m = 2T_{j'}T_n^{\beta_n-1}\otimes m,$$

so $x^{(\varepsilon_{j'}+\varepsilon_n)}T^\beta\otimes m = 0$ using 2.1 and the assumption $\beta_{j'} = p-1$.

If $\beta_j > 0$, then 1.1 and 2.1(3) give

$$\begin{aligned} x^{(\varepsilon_{j'}+\varepsilon_n)}T^\beta \otimes m &= T_j x^{(\varepsilon_{j'}+\varepsilon_n)}T^{\beta-\varepsilon_j}\otimes m + x^{(\varepsilon_n)}T^{\beta-\varepsilon_j}\otimes m \\ &\quad - x^{(\varepsilon_{j'}+\varepsilon_j)}T^{\beta-\varepsilon_j}\otimes m. \end{aligned}$$

Applying the induction hypothesis to the first term, 2.5 to the second term and 2.6 to the third term yields the desired formula. □

Lemma 2.10. *If $1 \le i \le r$ and* $\beta_k = \begin{cases} p-1, & 1 \le k < i \\ 0, & i < k < n \\ 1, & k = n, \end{cases}$ *then*

$$x^{(\varepsilon_{i'}+2\varepsilon_n)}T^\beta \otimes m = 2\beta_i(\lambda_n + \lambda_i - \beta_i + i)T^{\beta-\varepsilon_i-\varepsilon_n}\otimes m.$$

Proof. This can be proved by induction on β_i and the proof is similar to that of 2.9. (Here, 2.4 and 2.7 are required for the case $\beta_i > 0$.) □

Lemma 2.11. *Let $1 \le i \le r$, $r < j < n$ and $\mu, \omega \in \mathbf{Z}$. Then for any $v \in S(\lambda)$,*

$$\begin{aligned} x^{(\mu\varepsilon_{j'}+\omega\varepsilon_{i'})}T^\beta \otimes v = (-1)^\omega \binom{\beta_i}{\omega}\binom{\beta_j}{\mu-2} T^{\beta-\omega\varepsilon_i-(\mu-2)\varepsilon_j} \otimes x^{(2\varepsilon_{j'})}v \\ + (-1)^{\omega-1}\binom{\beta_i}{\omega-1}\binom{\beta_j}{\mu-1} T^{\beta-(\omega-1)\varepsilon_i-(\mu-1)\varepsilon_j} \otimes x^{(\varepsilon_{j'}+\varepsilon_{i'})}v \\ + (-1)^{\omega-2}\binom{\beta_i}{\omega-2}\binom{\beta_j}{\mu} T^{\beta-(\omega-2)\varepsilon_i-\mu\varepsilon_j} \otimes x^{(2\varepsilon_{i'})}v \\ + (-1)^{\omega}\binom{\beta_i}{\omega}\binom{\beta_j}{\mu-1} T^{\beta-\omega\varepsilon_i-(\mu-1)\varepsilon_j}T_{j'} \otimes v \\ + (-1)^{\omega-1}\binom{\beta_i}{\omega-1}\binom{\beta_j}{\mu} T^{\beta-(\omega-1)\varepsilon_i-\mu\varepsilon_j}T_{i'} \otimes v \\ + (-1)^{\omega}\binom{\beta_i}{\omega}\binom{\beta_j}{\mu} T^{\beta-\omega\varepsilon_i-\mu\varepsilon_j+\varepsilon_n} \otimes v. \end{aligned}$$

Proof. First assume $\beta_i = 0 = \beta_j$. Setting $a = \mu\varepsilon_{j'} + \omega\varepsilon_{i'}$ it follows from 2.1 that $x^{(a)}T^\beta \otimes v = T^\beta x^{(a)} \otimes v$. Therefore, if $\|a\| > 0$, then both sides of the formula are equal to zero. This leaves the cases $-2 \le \|a\| \le 0$ which are each routinely verified.

Now assume $\beta_i = 0$ and $\beta_j > 0$. Then

$$x^{(a)}T^\beta \otimes v = T^{\beta-\beta_j\varepsilon_j}x^{(a)}T_j^{\beta_j} \otimes v$$

and by 1.1,

$$x^{(a)}T_j^{\beta_j} \otimes v = T_j x^{(a)}T_j^{\beta_j-1} \otimes v + x^{(a-\varepsilon_{j'})}T_j^{\beta_j-1} \otimes v.$$

The induction hypothesis applies to these final two terms and the result follows.

Finally, if $\beta_i > 0$, then

$$x^{(a)}T^\beta \otimes v = T_i x^{(a)}T^{\beta-\varepsilon_i} \otimes v - x^{(a-\varepsilon_{i'})}T^{\beta-\varepsilon_i} \otimes v$$

and once again the formula is obtained by using the induction hypothesis. □

Lemma 2.12. *Assume $\beta_k = p-1$ for all $1 \le k \le r$ and for $k = n$. Set $M = u(K)T^\beta \otimes m$ and let $r < j < n$.*

In the following situations, $T^{\beta-\varepsilon_j} \otimes m \in M$.

1. $\lambda_{j'} \neq 0$ *and* $\beta_j = p-1$.

2. $\lambda_{j'} = 0$ *and* $\beta_j \neq p-1$.

3. $\lambda_{j'} = -1$ *and* $\beta_j \neq 1$.

4. *There exists* $j' < i \leq r$ *such that* $\lambda_i \neq 0$ *and* $\beta_{i'} = p-1$.

5. *There exists* $1 \leq i < j'$ *such that* $\lambda_i \neq -1$ *and* $\beta_{i'} = 0$.

In addition, the following statements hold.

6 *If* $\lambda_{j'} \neq -1$ *and* $\beta_j = p-2$, *then* $T^{\beta-\beta_j\varepsilon_j} \otimes m \in M$.

7 *Assume* $\beta_k = \begin{cases} 0, & r < k < j \\ p-1, & j \leq k < n. \end{cases}$ *If* $\lambda_{j'} = 0$ *and* $\lambda_n \neq j-3r-3$ *or if* $\lambda_{j'} = -1$ *and* $\lambda_n \neq j-3r-2$, *then* $T^{\beta-\beta_j\varepsilon_j} \otimes m \in M$.

Proof. (1–3) By 2.11,

$$x^{(2\varepsilon_{j'}+\varepsilon_j)}T^\beta \otimes m = (\beta_j/2)(2\lambda_{j'} + \beta_j + 1)T^{\beta-\varepsilon_j} \otimes m.$$

(Note that the first term on the right hand side in 2.11 is zero since m is maximal, the third term is zero since the first binomial coefficient is zero, the last term is zero since $\beta_n = p-1$, and the fourth and fifth terms simplify using 2.1(3).)

(4) By part (1), $T^{\beta-\varepsilon_{i'}} \otimes m \in M$, and by 2.11, $x^{(\varepsilon_{j'}+\varepsilon_{i'})}T^{\beta-\varepsilon_{i'}} \otimes m = \beta_j T^{\beta-\varepsilon_j} \otimes m$.

(5) By 2.11,

$$\begin{aligned} x^{(2\varepsilon_i+\varepsilon_{i'})}x^{(\varepsilon_{j'}+\varepsilon_{i'})}T^\beta \otimes m &= x^{(2\varepsilon_i+\varepsilon_{i'})}(T^\beta \otimes x^{(\varepsilon_{j'}+\varepsilon_{i'})}m + \beta_j T^{\beta+\varepsilon_{i'}-\varepsilon_j} \otimes m) \\ &= \beta_j(\lambda_i + 1)T^{\beta-\varepsilon_j} \otimes m. \end{aligned}$$

(6) By 2.11, $x^{((p-1)\varepsilon_{j'}+\varepsilon_j)}T^\beta \otimes m = (\lambda_{j'} + 1)T^{\beta-(p-2)\varepsilon_j} \otimes m$.

(7) First assume $\lambda_{j'} = 0$ and $\lambda_n \neq j-3r-3$. By 2.9,

$$x^{(\varepsilon_{j'}+\varepsilon_n)}T^\beta \otimes m = (j-3r-3-\lambda_n)T^{\beta-\varepsilon_j} \otimes m,$$

so $T^{\beta-\varepsilon_j} \otimes m \in M$. By part (2), $T^{\beta-\beta_j\varepsilon_j} \otimes m \in M$.

Now assume $\lambda_{j'} = -1$ and $\lambda_n \neq j-3r-2$. By part (3) it may be assumed that $\beta_j = 1$. By 2.9,

$$x^{(\varepsilon_{j'}+\varepsilon_n)}T^\beta \otimes m = (\lambda_n - j + 3r + 2)T^{\beta-\varepsilon_j} \otimes m,$$

and the proof is complete. □

Lemma 2.13. *Assume $\beta_k = 0$ for all $r < k < n$. Set $M = u(K)T^\beta \otimes m$ and let $1 \leq i \leq r$.*

In the following situations, $T^{\beta-\varepsilon_i} \otimes m \in M$.

1. *$\lambda_i \neq -1$ and $\beta_i = p-1$.*
2. *$\lambda_i = -1$ and $\beta_i \neq p-1$.*
3. *$\lambda_i = 0$ and $\beta_i \neq 1$.*
4. *There exists $r < j < i'$ such that $\lambda_{j'} \neq -1$ and $\beta_{j'} = p-1$.*
5. *There exists $i' < j < n$ such that $\lambda_{j'} \neq 0$ and $\beta_{j'} = 0$.*

In addition, the following statements hold.

6 If $\lambda_i \neq 0$ and $\beta_i = p-2$, then $T^{\beta-\beta_i\varepsilon_i} \otimes m \in M$.

7 Assume $\beta_k = \begin{cases} p-1, & 1 \leq k \leq i \\ 0, & i < k \leq r \\ 1, & k = n. \end{cases}$ If $\lambda_i = -1$ and $\lambda_n \neq -i$ or if $\lambda_i = 0$ and $\lambda_n \neq 1-i$, then $T^{\beta-\beta_i\varepsilon_i-\varepsilon_n} \otimes m \in M$.

Proof. (1–3) By 2.11,

$$x^{(\varepsilon_i+2\varepsilon_{i'})}T^\beta \otimes m = (\beta_i/2)(\beta_i - 2\lambda_i - 1)T^{\beta-\varepsilon_i} \otimes m.$$

(4) By part (1), $T^{\beta-\varepsilon_{j'}} \otimes m \in M$ and by 2.11, $x^{(\varepsilon_{j'}+\varepsilon_{i'})}T^{\beta-\varepsilon_{j'}} \otimes m = -\beta_i T^{\beta-\varepsilon_i} \otimes m$.

(5) By 2.11,

$$\begin{aligned} x^{(\varepsilon_{j'}+2\varepsilon_j)}x^{(\varepsilon_{j'}+\varepsilon_{i'})}T^\beta \otimes m &= x^{(\varepsilon_{j'}+2\varepsilon_j)}(T^\beta \otimes x^{(\varepsilon_{j'}+\varepsilon_{i'})}m - \beta_i T^{\beta+\varepsilon_{j'}-\varepsilon_i} \otimes m) \\ &= \beta_i\lambda_{j'}T^{\beta-\varepsilon_i} \otimes m. \end{aligned}$$

(6) By 2.11, $x^{(\varepsilon_i+(p-1)\varepsilon_{i'})}T^\beta \otimes m = -\lambda_i T^{\beta-(p-2)\varepsilon_i} \otimes m$.

(7) First assume $\lambda_i = -1$ and $\lambda_n \neq -i$. By 2.10,

$$x^{(\varepsilon_{i'}+2\varepsilon_n)}T^\beta \otimes m = -2(\lambda_n + i)T^{\beta-\varepsilon_i-\varepsilon_n} \otimes m,$$

so $T^{\beta-\varepsilon_i-\varepsilon_n} \otimes m \in M$. By part (2), $T^{\beta-\beta_i\varepsilon_i-\varepsilon_n} \otimes m \in M$.

Now assume $\lambda_i = 0$ and $\lambda_n \neq 1-i$. By part (3) it may be assumed that $\beta_i = 1$. By 2.10,

$$x^{(\varepsilon_{i'}+2\varepsilon_n)}T^\beta \otimes m = 2(\lambda_n - 1 + i)T^{\beta-\varepsilon_i-\varepsilon_n} \otimes m$$

and the proof is complete. □

Lemma 2.14. *Assume* $\beta_k = \begin{cases} p-1, & 1 \le k \le r \\ 0, & r < k < n \\ p-1, & k = n. \end{cases}$ *If* λ *is not exceptional, then* $T^{\beta-(p-2)\varepsilon_n} \otimes m \in M := u(K)T^{\beta} \otimes m$.

Proof. By 2.4,

$$x^{((p-1)\varepsilon_n)}T^{\beta} \otimes m = 2^{p-2}(\lambda_n + r + 2)T^{\beta-(p-2)\varepsilon_n} \otimes m,$$

so if $\lambda_n \neq -r-2$, the result follows. Assume $\lambda_n = -r-2$. Then $\sum_{i \neq n} \lambda_i \varepsilon_i \neq \zeta_1$ since λ is not exceptional. Therefore, either part (1) or part (4) of 2.13 applies to give $T^{\beta-\varepsilon_r} \otimes m \in M$. Again, by 2.4,

$$x^{(\varepsilon_r)}x^{((p-1)\varepsilon_n)}T^{\beta-\varepsilon_r} \otimes m = 2^{p-2}T^{\beta-(p-2)\varepsilon_n} \otimes m,$$

so the proof is complete. □

3 Proof of Theorem

The results of the previous section can now be assembled to prove the theorem.

Proof. (Proof of 1.2) Let N be a nonzero submodule of $M(S(\lambda))$ and let $0 \neq v \in N$. Write

$$v = \sum_{\beta \in A} c(\beta)T^{\beta} \otimes s_{\beta}$$

with $c(\beta) \in F$ and $s_{\beta} \in S(\lambda)$. Order A by setting $\beta < \beta'$ if for some k $(1 \le k \le n)$ $\beta_i = \beta'_i$ for all $i > k$ and $\beta_k < \beta'_k$. Let η be the least element for which $c(\eta) \neq 0$ and set $y = \prod_{i=1}^{n} T_i^{p-1-\eta_i}$. Then, using 2.1, $T^{\gamma} \otimes s_{\eta} = c(\eta)^{-1}yv \in N$ where $\gamma = \sum_{i=1}^{n}(p-1)\varepsilon_i$. Now $T^{\gamma} \otimes S(\lambda)$ is a $u(K_0)$-submodule of $M(S(\lambda))$; in fact, viewing $M(S(\lambda))$ as a graded module in the obvious way, it is the homogeneous component of least degree. Moreover, $T^{\gamma} \otimes S(\lambda)$ is simple so, since it intersects N nontrivially, it must be contained in N. Hence $T^{\gamma} \otimes m \in N$ for some maximal vector m of $S(\lambda)$.

For $0 \le i \le r$, $r < j \le n$ and $0 \le k < p$, set

$$[i, j, k] = \sum_{l=1}^{i}(p-1)\varepsilon_l + \sum_{l=j}^{2r}(p-1)\varepsilon_l + k\varepsilon_n$$

(empty sums being zero) and note that $\gamma = [r, r+1, p-1]$.

If $T^{[r,j,p-1]} \otimes m \in N$, then $T^{[r,j+1,p-1]} \otimes m \in N$ $(r < j < n)$. This follows from 2.12 by using parts (1) and (6) if $\lambda_{j'} \notin \{0, -1\}$, by using part (4) or (5) if $\lambda_{j'} \in \{0, -1\}$ and $\sum_{i \neq n} \lambda_i \varepsilon_i \notin \{\zeta_{2r-j+1}, \zeta_{2r-j+2}\}$, and by using part (7) and the assumption that λ is not exceptional otherwise. By induction, $T^{[r,n,p-1]} \otimes m \in N$.

Next, $T^{[r,n,1]} \otimes m \in N$ by 2.14.

Finally, if $T^{[i,n,1]} \otimes m \in N$, then $T^{[i-1,n,0]} \otimes m \in N$ $(0 < i \leq r)$ (and consequently $T^{[i-1,n,1]} \otimes m = T_n T^{[i-1,n,0]} \otimes m \in N$). This follows from part (7) of 2.13 if $\sum_{i \neq n} \lambda_i \varepsilon_i \in \{\zeta_{r-i+1}, \zeta_{r-i+2}\}$, so suppose otherwise. Assume $\lambda_n \neq -i$. By 2.4, $x^{(2\varepsilon_n)} T^{[i,n,1]} \otimes m = -2(\lambda_n + i) T^{[i,n,0]} \otimes m$, so that $T^{[i,n,0]} \otimes m \in N$. Then, from 2.13 it follows that $T^{[i-1,n,0]} \otimes m \in N$ by using parts (1) and (6) if $\lambda_i \notin \{0, -1\}$ and by using part (4) or (5) otherwise. Now assume $\lambda_n = -i$. By 2.13, using part (1) if $\lambda_i \neq -1$ and part (4) or (5) otherwise, one finds that $T^{[i,n,0]-\varepsilon_i} \otimes m \in N$. Now 2.4 gives $x^{(2\varepsilon_n)} T^{[i,n,1]-\varepsilon_i} \otimes m = -2T^{[i,n,0]-\varepsilon_i} \otimes m \in N$ so that $T^{[i,n,0]-\varepsilon_i} \otimes m \in N$. As before, 2.13 implies that $T^{[i-1,n,0]} \otimes m \in N$, here using part (6) if $\lambda_i \neq 0$ and part (4) or (5) otherwise.

By induction, $1 \otimes m = T^{[0,n,0]} \otimes m \in N$ and, since this vector generates $M(S(\lambda))$, it follows that $N = M(S(\lambda))$. This completes the proof. □

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